A Recoverable Annular Network with Unrecoverable Dual

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Recovering the Network

For definitions, see [1]. I consider the network Γ shown in Figure 1 and its dual $\Gamma^\dagger$ shown in Figure 3. I show that Γ is recoverable, while for $\Gamma^\dagger$, the inverse problem has a two-parameter family of solutions.

To recover Γ, we use the boundary value problem shown in Figure 2. We first determine the conductance of the spike $\gamma(5, 13)$. By a symmetrical argument, we can recover the other spikes. Then the same boundary value problem allows us to determine $\gamma(4, 13)$. Knowing the current on $4 \to 13$ and net current on 4, we can determine the current on $4 \to 12$ and hence $\gamma(4, 12)$. By symmetry, we can determine $\gamma(1, 9)$, $\gamma(3, 11)$, and $\gamma(6, 14)$. Then we know the voltage on 11 and 15, so we can recover $\gamma(11, 12)$ and $\gamma(11, 15)$. By symmetry, we can recover all the remaining edges.

Figure 1: The network Γ.

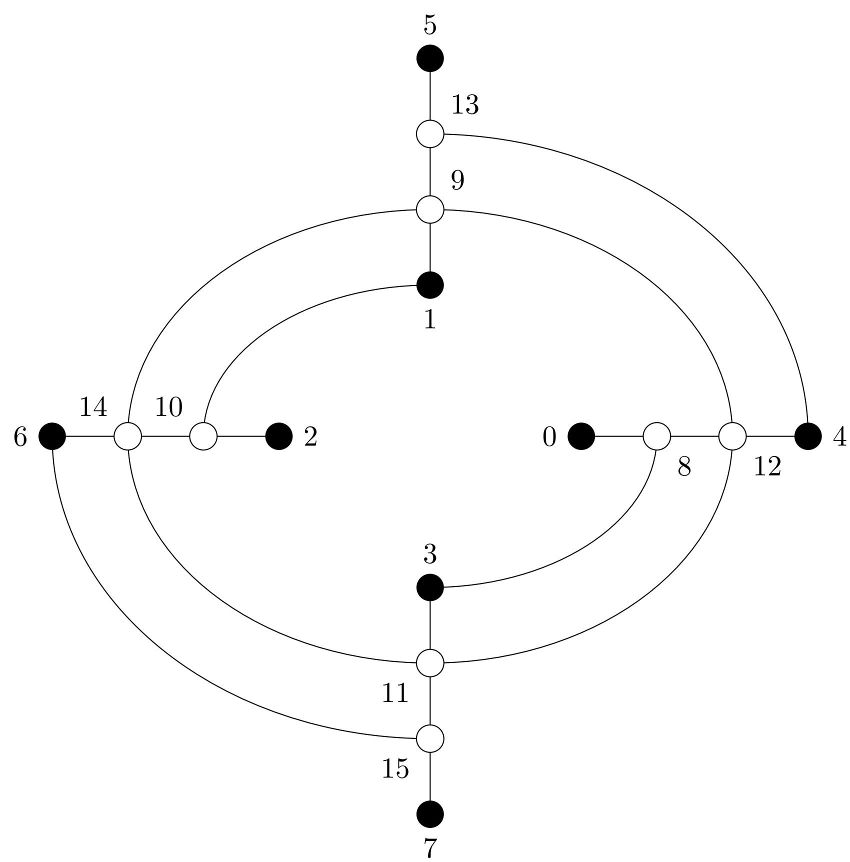


Figure 2: Boundary value problem to recover conductances.

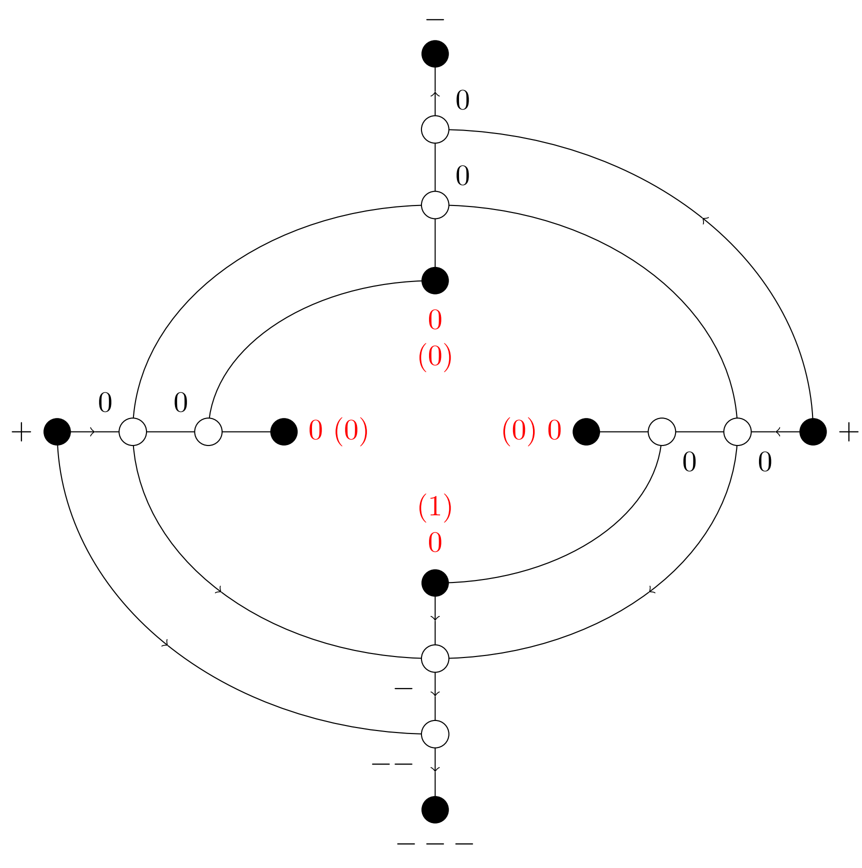
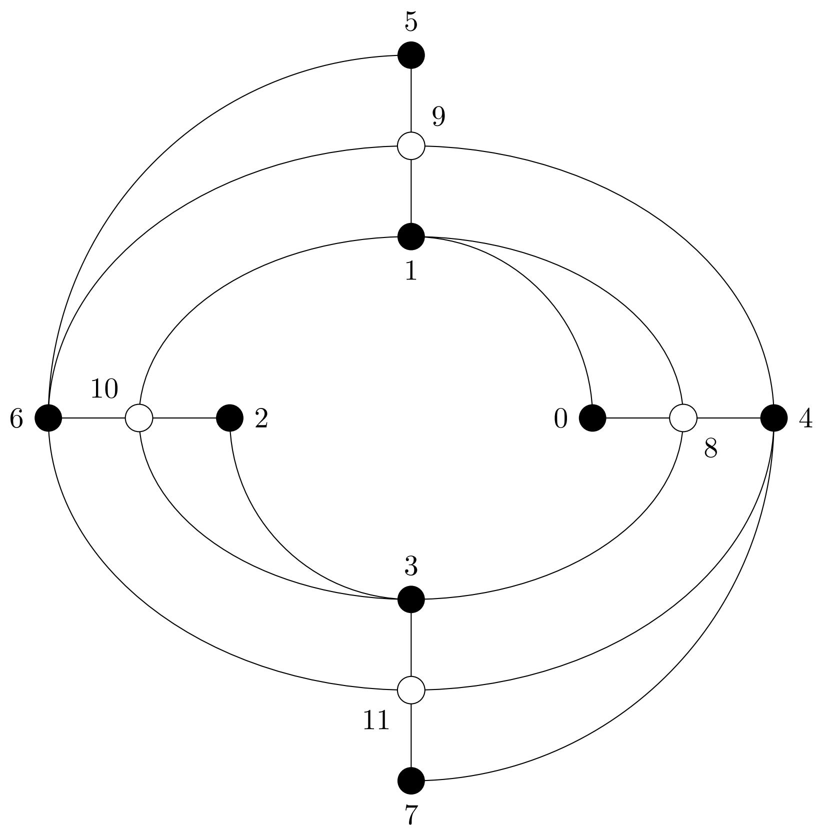


Figure 3: The network $\Gamma^\dagger$.



The Unrecoverable Dual

The dual graph is shown in Figure 3. For any valid response matrix, the inverse problem has a two-parameter family of solutions in a neighborhood of the original conductivity function.

To see this, we use the $\bigstar$-$\mathcal{K}$ transformation described in [2]. We replace each four-star in the network with an electrically equivalent complete graph on four vertices. See Figure 4. The conductances on each $\mathcal{K}_4$ satisfy the quadrilateral rule. Conversely, any complete graph whose conductances satisfy the quadrilateral rule is electrically equivalent to a star.

A conductivity function γ on the original graph produces a conductivity function on the transformed graph μ. The response matrix of the transformed graph is easy to compute: To find λ_{pq}, simply add up the conductances of the edges from p to q. To refer to an edge which is part of a $\mathcal{K}_4$, I will list three numbers. The first and last will be the vertices at the endpoints of the edge and the middle one will be the name of the interior vertex which is the center of the star corresponding to the $\mathcal{K}_4$. For instance, the conductance of the blue edge joining 4 and 6 is $\mu(4, 9, 6)$. The boundary edges of the original graph are shown in black and their conductances will be called simply $\mu(0, 1)$, $\mu(2, 3)$, $\mu(4, 7)$, and $\mu(5, 6)$.

We construct a new conductivity function μ_{xy} depending on two parameters x and y, such that μ_{xy} satisfies the quadrilateral rule and has the same response matrix as μ. To begin with, μ_{xy} must be the same as μ for all singleton edges:

$$\begin{array}{rr}
\mu_{xy}(0,8,4) = \mu(0,8,4) & \mu_{xy}(4,9,5) = \mu(4,9,5) \\
\mu_{xy}(1,9,5) = \mu(1,9,5) & \mu_{xy}(6,11,7) = \mu(6,11,7) \\
\mu_{xy}(2,10,6) = \mu(2,10,6) & \mu_{xy}(0,8,3) = \mu(0,8,3) \\
\mu_{xy}(3,11,7) = \mu(3,11,7) & \mu_{xy}(1,10,2) = \mu(1,10,2).
\end{array}$$

We do not include $\mu(4, 9, 6)$, etc., because there are two edges from 4 to 6, one above the hole and one below it. Instead, we define

$$\begin{array}{rr}
\mu_{xy}(4,9,6) = \mu(4,9,6) + x & \mu_{xy}(4,11,6) = \mu(4,11,6) - x, \\
\mu_{xy}(1,8,3) = \mu(1,8,3) + y & \mu_{xy}(1,10,3) = \mu(1,10,3) - y.
\end{array}$$

This way,

$$\mu_{xy}(4,9,6) + \mu_{xy}(4,11,6) = \mu(4,9,6) + \mu(4,11,6) = \lambda(4,6),$$

and similarly for the other pair.

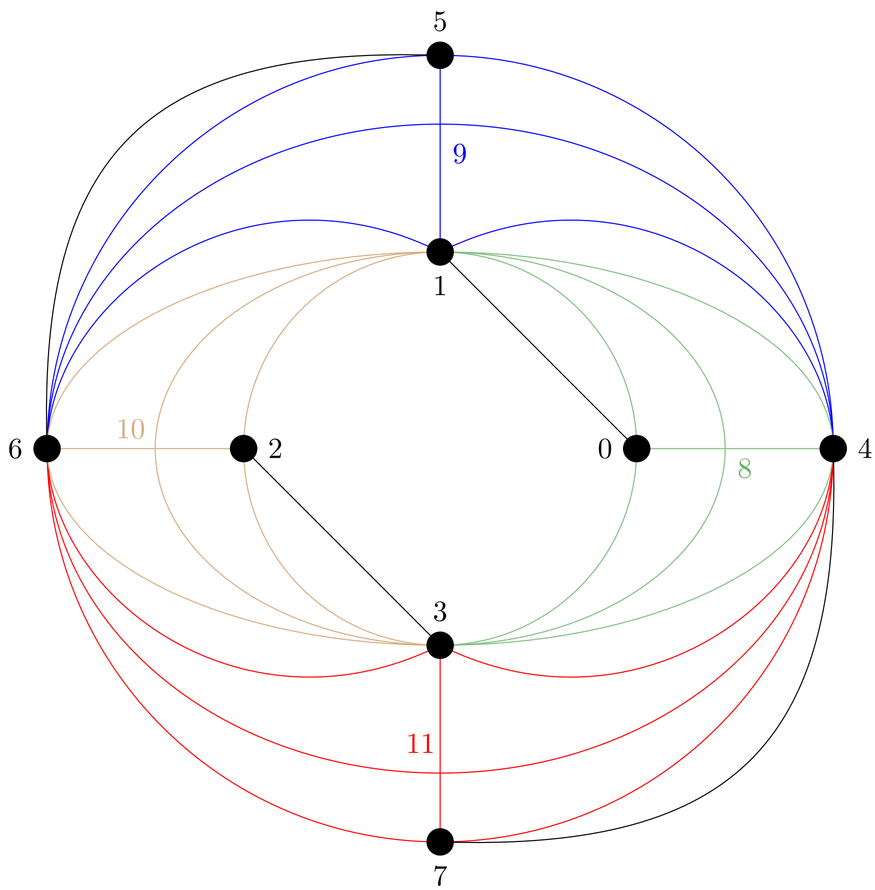
We find the other values of μ_{xy} using the quadrilateral rule. For instance, applying the quadrilateral rule to the $\mathcal{K}_4$ corresponding to vertex 9 (blue) gives

$$\mu_{xy}(1,9,6) = \frac{\mu_{xy}(1,9,5)\mu_{xy}(4,9,6)}{\mu_{xy}(4,9,5)} = \mu(1,9,6) + \frac{\mu(1,9,5)}{\mu(4,9,5)}x.$$

Since we want $\mu_{xy}(1, 9, 6) + \mu_{xy}(1, 10, 6) = \mu(1, 9, 6) + \mu(1, 6, 10)$, we let

$$\mu_{xy}(1,10,6) = \mu(1,10,6) - \frac{\mu(1,9,5)}{\mu(4,9,5)}x.$$

Figure 4: $\Gamma^{\dagger}$ after ★-$\mathcal{K}$ transformation.



By a symmetrical argument, we let

$$\begin{aligned}
\mu_{xy}(3,11,4) &= \mu(3,11,4) - \frac{\mu(3,11,7)}{\mu(4,11,6)}x, \\
\mu_{xy}(3,8,4) &= \mu(3,8,4) + \frac{\mu(3,11,7)}{\mu(4,11,6)}x, \\
\mu_{xy}(1,8,4) &= \mu(1,8,4) + \frac{\mu(0,8,4)}{\mu(0,8,3)}y, \\
\mu_{xy}(1,9,4) &= \mu(1,9,4) - \frac{\mu(0,8,4)}{\mu(0,8,3)}y, \\
\mu_{xy}(3,10,6) &= \mu(3,10,6) - \frac{\mu(2,10,6)}{\mu(1,10,2)}y, \\
\mu_{xy}(3,11,6) &= \mu(3,11,6) + \frac{\mu(2,10,6)}{\mu(1,10,2)}y.
\end{aligned}$$

Finally, we determine $\mu_{xy}(0,8,1)$:

$$\begin{aligned}
\mu_{xy}(0,8,1) &= \frac{\mu_{xy}(1,8,3)\mu_{xy}(0,8,4)}{\mu_{xy}(3,8,4)} \\
&= \frac{\mu(0,8,4)(\mu(1,8,3)+y)}{\mu(3,8,4)+\frac{\mu(3,11,7)}{\mu(4,11,6)}x}.
\end{aligned}$$

Then let

$$\mu_{xy}(0,1) = \lambda(0,1) - \mu_{xy}(0,8,1).$$

We determine $\mu(5,9,6)$ and $\mu(5,6)$, $\mu(2,10,3)$ and $\mu(2,3)$, $\mu(4,11,7)$ and $\mu(4,7)$ in a similar way.

The resulting conductivity function μ_{xy} satisfies the quadrilateral rule at each of the four-stars, and it has the same response matrix as μ. To show that the graph is unrecoverable, we only need to show that μ_{xy} has positive conductances for some nonzero (x,y). For $(x,y)=(0,0)$, μ_{xy} is exactly μ, so all conductances are positive there. Since μ_{xy} is a continuous function of (x,y), all conductances will be positive in a neighborhood of $(0,0)$.

Since μ_{xy} satisfies the quadrilateral rule, it corresponds to a conductivity function γ_{xy} on the original graph, which will be positive if μ_{xy} is positive, and depends continuously on μ_{xy}. Thus, the inverse problem has a two-parameter family of solutions in a neighborhood of the original γ. We know we cannot add any more parameters because after we added the initial parameters x and y, all other values were determined from the quadrilateral rule and the response matrix.

Conclusion

A recoverable network with unrecoverable dual is a counterexample to some hopeful conjectures about annular networks, showing that

- A lensless medial graph does not always imply recoverability (see Figure 5).
- The medial graph, without a coloring of the cells, is insufficient to determine recoverability.
- It is impossible in general to determine the response matrix of the dual graph from the response matrix of the primal graph.
- In circular planar networks, two problems which are "dual" are equally difficult, but in annular networks, one is harder than the other.

Appendix: Recovering $\Gamma^{\dagger}$ when Two Conductances are Known

Suppose we know the conductances of edges $\gamma(0,1)$ and $\gamma(4,7)$. Suppose we delete these edges and update the response matrix. Then we use boundary value problem in Figure 6 to find the conductance of the spike $\gamma(7,11)$. Next, we use the problem in Figure 7 to find $\gamma(0,8)$. We know that we can find a and b that will make the currents at 4 and 7 be 0 and 1 because $\Lambda(0,3;4,7)$ is invertible; there is only one connection between 0, 3 and 4, 7. We will have b negative and a positive. Once we find the current at 0 we can find $\gamma(0,8)$.

After we contract the spikes, the conductances of the boundary edges on the lower right can be read directly from the response matrix. Once we delete those edges, the graph becomes circular planar and recovering the rest of the conductances is straightforward.

Knowing how to recover the graph with two edges deleted, we can compute two sets of conductances with the same response matrix. We begin by setting all conductances to 1 and computing the response matrix. Then we create a new Kirchhoff matrix. We set $\gamma(0,1)$ and $\gamma(4,7)$ to *different* values from 1. Then we use the process outlined above to "recover" the other conductances. From the new Kirchhoff matrix, we compute a new response matrix–which always turns out to be the same as the original response matrix. However, to make the new conductances positive, we have to choose our parameters close enough to 1.

Figure 5: The medial graph of Γ. Notice it is completely lensless.

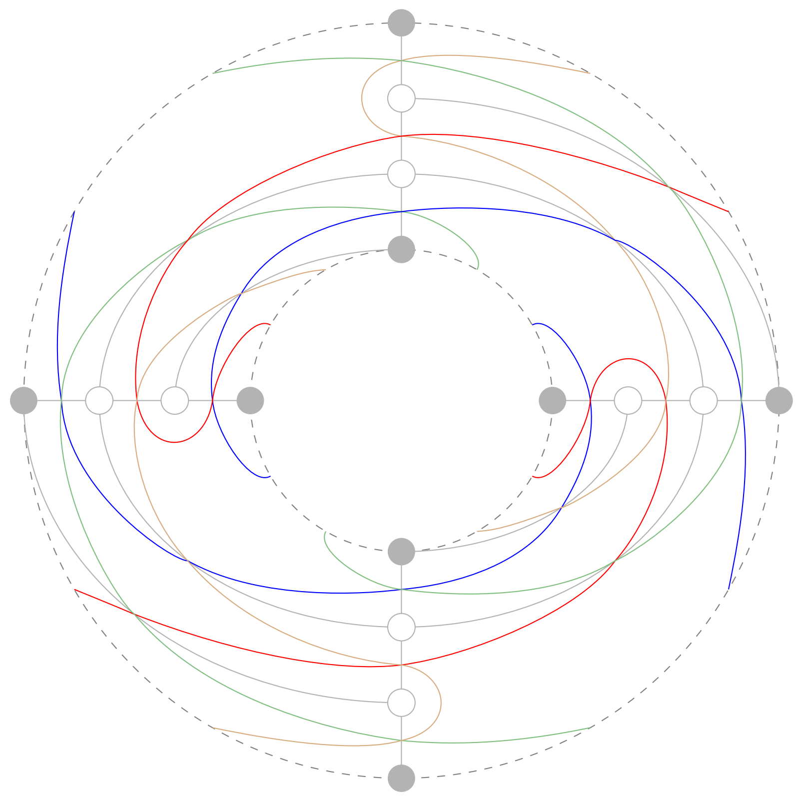


Figure 6: Boundary value problem to find $\gamma(7, 11)$.

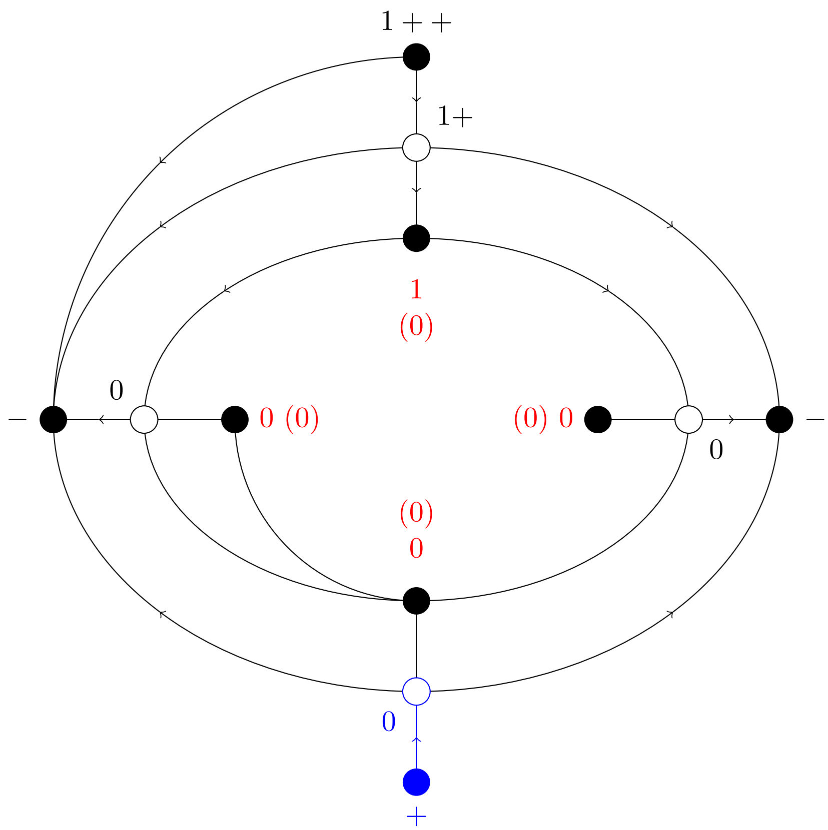
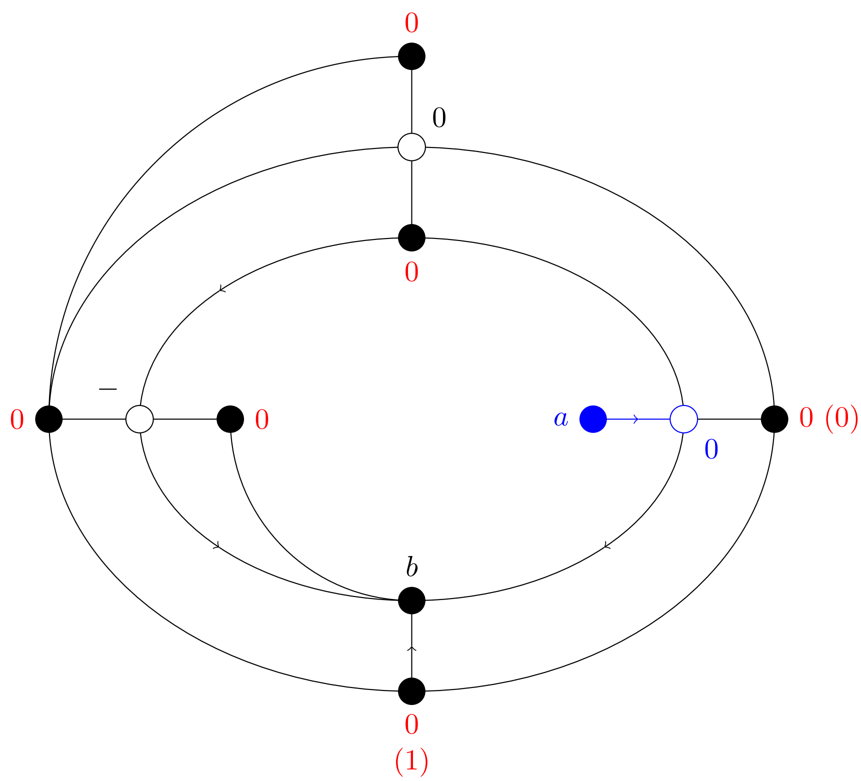


Figure 7: Boundary value problem to find $\gamma(0,8)$.



The Sage code below implements this process.

```
#these are the parameters for the edges 0 to 1 and 4 to 7
x = 1.0
y = 1.0

#updates response matrix after deleting boundary edge
def deleteboundaryedge(L,i,j,c):
     L[i,j] += c
     L[j,i] += c
     L[i,i] -= c
     L[j,j] -= c
     return L

#updates response matrix after contracting spike
def contractspike(L,i,c): #update response matrix
     d = L[i,i] - c
     A = L[i,:]
     A[0,i] = 0
     L = L - A.transpose() * A / d
     L[i,:] = -c * A[0,:] / d
     L[:,i] = -c * A.transpose()[:,0] / d
     L[i,i] = -c * (1 + c / d)
     return L

#original Kirchhoff matrix with all conductances set to 1
K = matrix(RR,
    [[2,-1,0,0,0,0,0,0,-1,0,0,0],
    [-1,4,0,0,0,0,0,0,-1,-1,-1,0],
    [0,0,2,-1,0,0,0,0,0,0,-1,0],
    [0,0,-1,4,0,0,0,0,-1,0,-1,-1],
    [0,0,0,0,4,0,0,-1,-1,-1,0,-1],
    [0,0,0,0,0,2,-1,0,0,-1,0,0],
    [0,0,0,0,0,-1,4,0,0,-1,-1,-1],
    [0,0,0,0,-1,0,0,2,0,0,0,-1],
    [-1,-1,0,-1,-1,0,0,0,4,0,0,0],
    [0,-1,0,0,-1,-1,-1,0,0,4,0,0],
    [0,-1,-1,-1,0,0,-1,0,0,0,4,0],
    [0,0,0,-1,-1,0,-1,-1,0,0,0,4]])
```

```
#original response matrix
L = K[0:8,0:8] - K[0:8,8:12] * K[8:12,8:12].inverse() * K[8:12,0:8]

print 'original Kirchhoff matrix\n', K
print 'original response matrix\n', L

#new Kirchhoff matrix to store the new values
newK = matrix(RR,12,12)

newK[0,1] = -x
newK[4,7] = -y

L = deleteboundaryedge(L,0,1,x)
L = deleteboundaryedge(L,4,7,y)

#first spike
P = -L[0:4,4:8].inverse() * L[0:4,0:4]
Q = L[4:8,0:4] + L[4:8,4:8] * P
cond = Q[3,1] / P[3,1]
newK[7,11] = -cond
print cond
L = contractspike(L,7,cond)

#second spike
S = matrix([[L[4,0],L[4,3]],[L[7,0],L[7,3]]]).inverse()
a = S[0,1]
b = S[1,1]
currentA = a * L[0,0] + b * L[0,3]
cond = currentA / a
newK[0,8] = -cond
L = contractspike(L,0,cond)

#boundary edges
newK[4,11] = L[4,7]
L = deleteboundaryedge(L,4,7,-L[4,7])

newK[3,8] = L[0,3]
L = deleteboundaryedge(L,0,3,-L[0,3])
```

```
newK[3,11] = L[3,7]
L = deleteboundaryedge(L,3,7,-L[3,7])

newK[6,11] = L[6,7]
L = deleteboundaryedge(L,6,7,-L[6,7])

newK[4,8] = L[0,4]
L = deleteboundaryedge(L,0,4,-L[0,4])

newK[1,8] = L[0,1]
L = deleteboundaryedge(L,0,1,-L[0,1])

#third spike
a = -L[4,5] / L[1,5]
cond = a * L[1,4] + L[4,4]
newK[4,9] = -cond
L = contractspike(L,4,cond)

#more boundary  edges
newK[1,9] = L[1,4]
L = deleteboundaryedge(L,1,4,-L[1,4])

newK[5,9] = L[4,5]
L = deleteboundaryedge(L,4,5,-L[4,5])

newK[6,9] = L[4,6]
L = deleteboundaryedge(L,4,6,-L[4,6])

newK[5,6] = L[5,6]
L = deleteboundaryedge(L,5,6,-L[5,6])

#fourth spike
a = -L[1,6] / L[2,6]
cond = a * L[1,2] + L[1,1]
newK[1,10] = -cond
L = contractspike(L,1,cond)

#last boundary edges
newK[2,10] = L[1,2]
newK[3,10] = L[1,3]
```

```
newK[6,10] = L[1,6]
newK[2,3] = L[2,3]

#fill in newK below the diagonal
n = len(newK.column(0))
for i in range(n):
     for j in range(i):
          newK[i,j] = newK[j,i]
for i in range(n):
     sum = 0
     for j in range(n):
          sum += newK[i,j]
     newK[i,i] = -sum

#compute new response matrix
newL = newK[0:8,0:8]
     - newK[0:8,8:12] * newK[8:12,8:12].inverse() * newK[8:12,0:8]

print 'new Kirchhoff matrix\n', newK
print 'new response matrix\n', newL
```

References

[1] Edward B. Curtis and James A. Morrow. *Inverse Problems for Electrical Networks.* World Scientific. 2000.

[2] Jeff Russell. "★ and $\mathcal{K}$ Solve the Inverse Problem."